MMP Learning Seminar.

Week 70

Content:

Boundedness of varieties of general type.

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Theorem (DCC of volumes): Fix $n \in \mathbb{N}$ and a set $I \subseteq [0,1]$ which satisfies the DCC.

Let $\mathfrak{D}$ be the set of projective $\boxed{\text{log canonical pairs}}$ (X, B) such that $\dim X = n$, and $\boxed{\operatorname{coeff}(B) \subseteq I.}$

Then there is a constant $\delta > 0$ and a positive integer m such that:

(1) The set $\{ \operatorname{vol}(X, K_X + B) \mid (X, B) \in \mathfrak{D} \}$ satisfies the DCC.

(2) if $\operatorname{vol}(X, K_X + B) > 0$, then $\operatorname{vol}(X, K_X + B) \geq \delta$.

(3) if $K_X + B$ is big, then $\phi_{m(K_X + B)}$ is birational.

$$\begin{array}{c} Y \\ \Big\downarrow{\scriptstyle \varphi} \\ X \end{array}$$

$\varphi^*(K_X) = K_Y + (1+\varepsilon) E$

This ε could go to zero

Theorem (Boundedness of anti-canonical volumes):

Let $\mathcal{D}$ be the set of $\boxed{\text{klt pairs}}$ (X, B) such that

X is projective, $\dim X = n$.

$K_X + B \equiv 0$, and $\boxed{\operatorname{coeff}(B) \subseteq I}$

Then, there exists a constant $M \geq 0$ only depending on n and I such that $\operatorname{vol}(X, -K_X) < M$ for every pair $(X, B) \in \mathcal{D}$.

Example: $X = \mathbb{P}(a_0, \dots, a_n)$, (X, E)

Assume well-formed.

$\uparrow$ log canonical CY

$K_X + E \sim 0$.

$$\operatorname{Vol}(X, -K_X) = \frac{(a_0 + \dots + a_n)^n}{(a_0 \cdots a_n)}.$$

$$\boxed{\begin{array}{c} |-MK_X| \ni \Gamma \\ (X, \Gamma/M) \end{array}}$$

$X_p = \mathbb{P}(p, 3, 2)$, $\frac{(p+5)^2}{6p} \longrightarrow \infty$. if $p \to \infty$.

Homework: Find the minimum m s.t $|-mK_X|$ admits a klt element. **Ans:** $m \sim \sqrt{p}$.

Theorem (Effective birationality):

$\mathcal{B}$ the set of lc pairs (X, B) such that X is projective, $\dim X = n$, $K_X + B$ is big. $\mathrm{coeff}(B) \subseteq I$.

Then $\phi_{m(K_X+B)}$ is birational where $m := m(n, I)$.

Theorem (The ACC for numerically trivial pairs):

There exists a finite subset $I_0 \subseteq I$ such that if (X, B) satisfies the following:

(1) (X, B) is an n-dimensional projective $\boxed{\text{lc pair}}$.

(2) $\boxed{\mathrm{coeff}(B) \subseteq I}$, and

(3) $K_X + B \equiv 0$.

Then, the coefficient sets of B belong to I_0

Theorem (The ACC for log canonical thresholds):

There exists a constant $\delta > 0$ such that if:

(1) (X, B) is a n-dimensional log pair with $\text{coeff}(B) \subseteq I$.

(2) (X, Φ) is klt for some $\Phi \geq 0$, and

(3) $B' \geq (1-\delta)B$ where (X, B') is log canonical.

Then (X, B) is log canonical.

used to tie-breaking.

- Boundedness of anti-canonical volumes:

(X,B) klt, $K_X + B \equiv 0$, and $\mathrm{vol}(-K_X) > 0$.

is really large.

$0 \leq G \sim_{\mathbb{Q}} -K_X$,

$$\mathrm{mult}_x(G) > \frac{1}{2}\left(\mathrm{vol}(X, -K_X)\right)^{\frac{1}{n}}.$$

(X, tG) is log canonical for t very small. not klt.

(X, B) klt.

$(X, \Phi = (1-\delta)B + \delta G)$ with the smallest real number for which the previous pair is lc but not klt.

coeff going to one.

$$K_X + \Phi \equiv (1-\delta)K_X + B.$$

The rest of the proof consists of a global-to-local argument and show that $\delta > 0$ small violates ACC.

Theorem (Boundedness of varieties of general type): Fix $n \in \mathbb{N}$ and a set $I \subseteq [0,1] \cap \mathbb{Q}$ satisfying the DCC & $d > 0$. Then, the set $\mathfrak{F}_{slc}(n, I, d)$ is bounded, that is, there exists a projective morphism of q.p varieties $\pi : \mathcal{X} \longrightarrow T$ and a $\mathbb{Q}$-divisor $\mathcal{B}$ on $\mathcal{X}$ such that the set of pairs $\{(\mathcal{X}_t, \mathcal{B}_t) \mid t \in T\}$ given by the fibers of π is in bijection with the elements of $\mathfrak{F}_{slc}(n, I, d)$.

Last step: $X^\nu \longrightarrow X$, $X^\nu = \bigsqcup X_i$.

$K_{X^\nu} + B^\nu$ is ample.

$K_{X^\nu} + B^\nu|_{S^\nu}$ is ample $\Longrightarrow$ involution τ belongs to an algebraic group.

(X, B, S, τ) is bounded.

$\mathrm{Diff}_{S^\nu}(B^\nu) \longleftarrow \tau$ must fix this.

Proposition 4.1: Fix $w \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$, I satisfying the DCC.

(Z, D) projective log smooth n-dimensional variety. D reduced

M_D = strict transform of D + reduced exceptional

↑ b-divisor.

There exists a finite sequence of blow-ups $Z' \longrightarrow Z$ of strata of M_D.

such that if:

(1) (X, B) is proj log smooth n-dim,

(2) $g : X \longrightarrow Z$ is a finite sequence of blow-ups of strata of M_D.

(3) $\operatorname{coeff}(B) \subseteq I$.

(4) $g_* B \leq D$, and $\boxed{B \leq M_{D,X}}$

(5) $\operatorname{Vol}(X, K_X + B) = w$.

Then, $\operatorname{vol}(Z', K_{Z'} + M_{B,Z'}) = w$.

$Z' \dashrightarrow X$

Proposition 4.2: Fix $n \in \mathbb{N}$, $d \geq 0$ and $I \subseteq [0,1] \cap \mathbb{Q}$ satisfying the DCC. Let $\mathfrak{F}_{lc}(n,d,I)$ be the set of pairs (X,B) which are disjoint union of ample models (X_i, B_i) where $\dim X_i = n$, $\operatorname{coeff}(B_i) \subseteq I$ and $(K_X + B)^n = d$. Then, $\mathfrak{F}_{lc}(d,I,n)$ is bounded.

Proof: Assume irreducible & consider (X_i, B_i), we have a log birationally bounded family:

$$\begin{tikzcd}
(Z,D) \arrow[r] & T \\
(Z',D') \arrow[u] \arrow[ur, "\text{proposition 4.1}"'] &
\end{tikzcd}$$

(Z', Φ) is a terminal pair.

Use invariance of plurigenera to prove that.

$$\operatorname{Vol}(Z'_{t_i}, K_{Z'_{t_i}} + \Phi_{t_i}) = d \quad \text{, constant.}$$

The ample model of the Z'_{t_i} are just X_i. $\square$

Lemma 4.3: (X, B) lc pair $K_X + B$ is big.

$f : X \dashrightarrow W$ an ample model for $K_X + B$

If $B' \geq B$, (X, B') is lc and $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_X + B')$.

Then W is also an ample model for $K_X + B'$.

Proof: $f : X \longrightarrow W$ is a morphism. $A = f_*(K_X + B)$

$F := K_X + B - f^*A$ is effective & f-exceptional.

$$\left.\begin{aligned} \mathrm{vol}(X, K_X + B) &= \mathrm{vol}(X, K_X + B + t(B' - B)) \\ &\geq \mathrm{vol}(X, f^*A + t(B' - B)) \\ &\geq \mathrm{vol}(X, f^*A) \\ &= \mathrm{vol}(X, K_X + B) \end{aligned}\right\} \text{independent of } t$$

E a component of $B' - B$.

positivity

$$0 = \frac{d}{dt} \mathrm{vol}(X, f^*A + tE))|_{t=0} = n \, \mathrm{vol}_E(f^*A)$$
$$\geq n \cdot E \cdot f^*A^{n-1}$$
$$= n \deg f_* E$$

Hence E is f-exceptional.

$$H^0(X, \mathcal{O}_X(m(K_X+B'))) =$$

$$H^0(X, \mathcal{O}_X(mf^*A + m(E+F))) =$$

$$H^0(X, \mathcal{O}_X(mf^*A)) =$$

$$H^0(X, \mathcal{O}_X(m(K_X+B))).$$

K_X+B & K_X+B' have the same canonical ring $\square$.